

THE PROBLEM OF DEFICIENCY INDICES ADJACENCY MATRICES ON LOCALLY FINITE GRAPHS

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ABSTRACT. In this note we answer negatively to our conjecture concerning the deficiency indices. More precisely, given any non-negative integer n , there is locally finite graph on which the adjacency matrix has deficiency indices (n, n) .

Let V be a countable set. Let $E := V \times V \rightarrow \{0, 1\}$ and assume that $E(x, y) = E(y, x)$, for all $x, y \in V$. Set that $E(x, x) = 0$, for all $x \in V$. We say that $x, y \in V$ are *neighbors* if $E(x, y) \neq 0$ and denote it by $x \sim y$. We say that $G := (E, V)$ is an unoriented simple graph with *vertices* V , *weights* E , and no loop. We associate to G the complex Hilbert space $\ell^2(V)$. We denote by $\langle \cdot, \cdot \rangle$ and by $\|\cdot\|$ the scalar product and the associated norm, respectively. The set of complex functions with compact support in V is denoted by $\mathcal{C}_c(G)$. The *adjacency matrix* is defined by:

$$(1) \quad (\mathcal{A}_G, \circ f)(x) := \sum_{y \sim x} f(y), \text{ with } f \in \mathcal{C}_c(G).$$

The operator is symmetric and thus closable. We denote the closures by \mathcal{A}_G . We denote the domains by $\mathcal{D}(\mathcal{A}_G)$, and its adjoint by $(\mathcal{A}_G)^*$. Even in the case of a locally finite tree G , \mathcal{A}_G may have many self-adjoint extensions. We investigate the number of possible self-adjoint extensions of the adjacency matrix by computing its deficiency indices. Given a closed and densely defined symmetric operator T acting on a complex Hilbert space, the deficiency indices of T are defined by $\eta_{\pm}(T) := \dim \ker(T^* \mp i) \in \mathbb{N} \cup \{+\infty\}$. The operator T possesses a self-adjoint extension if and only if $\eta_+(T) = \eta_-(T)$. If this is the case, we denote the common value by $\eta(T)$. T is self-adjoint if and only if $\eta(T) = 0$. Since the operator \mathcal{A}_G commutes with the complex conjugation, its deficiency indices are equal. This means that \mathcal{A}_G possesses a self-adjoint extension. Note that $\eta(\mathcal{A}_G) = 0$ if and only if \mathcal{A}_G is essentially self-adjoint on $\mathcal{C}_c(G)$.

In [MO, Mü], one constructs adjacency matrices for simple trees with positive deficiency indices. In fact, it follows from the proof that the deficiency indices are infinite in both references. As a general result, a special case of [GS, Theorem 1.1] gives that, given a locally finite simple tree G , one has the following alternative:

$$(2) \quad \eta(\mathcal{A}_G) \in \{0, +\infty\}.$$

The value of $\eta(\mathcal{A}_G)$ is discussed in [GS] and is linked with the growth of the tree.

In [MW, Section 3], one finds:

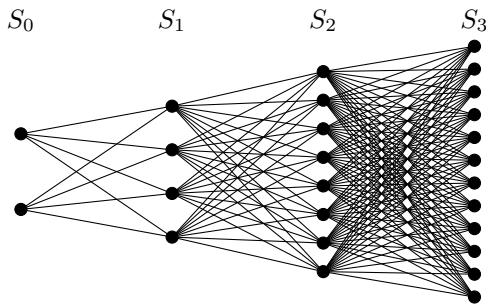
Theorem 1. *For all $n \in \mathbb{N} \cup \{\infty\}$, there is a simple graph G , such that $\eta(\mathcal{A}_G) = n$.*

Their proof is unfortunately incomplete. However, the statement is correct, this is aim of this note. Using standard perturbation theory, e.g., [GS, Proposition A.1], one sees that the validity of Theorem 1 is equivalent to the existence of a simple graph G for which

$$(3) \quad \eta(\mathcal{A}_G) = 1.$$

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FIGURE 1. An antitree with $s_0 = 2$, $s_1 = 4$, $s_2 = 8$, and $s_3 = 12$

In [MW], instead of pointing a simple graph such that (3) holds, they relied on a tree. More precisely, they refer to the works of [MO, Mü].

Keeping that in mind and strongly motivated by some other examples, we had proposed a drastically different scenario and had conjectured in [GS] that for any simple graph, one has (2).

We now turn to the proof of Theorem 1 and therefore disprove our conjecture. We rely on the decomposition of anti-trees, see [BK, Theorem 4.1].

Let S_n , $n \in \mathbb{N}$, be nonempty, finite and pairwise disjoint sets. We set $s_n := |S_n|$, $V := \cup_{n \in \mathbb{N}} S_n$ and $|x| := n$ for $x \in S_n$. Set also $E(x, y) = 1$, if $||x| - |y|| = 1$, and $E(x, y) = 0$ otherwise. We define:

$$P := \bigoplus_{n \in \mathbb{N}} P_n, \text{ where } P_n f(x) := \frac{1}{s_n} \mathbf{1}_{S_n}(x) \sum_{y \in S_n} f(y),$$

for all $f \in \ell^2(V)$. Note that $P = P^2 = P^*$ and $\text{rank } P \mathbf{1}_{S_n} = 1$ for all $n \in \mathbb{N}$. Given $f \in \ell^2(V)$ such that $f = Pf$, i.e., f is radially symmetric, we set $\tilde{f}(|x|) := f(x)$, for all $x \in V$. Note that

$$P\ell^2(V) = \{f \in \ell^2(V), \sum_{n \in \mathbb{N}} s_n |\tilde{f}(n)|^2 < \infty\} \simeq \ell^2(\mathbb{N}, (s_n)_{n \in \mathbb{N}}),$$

where $(s_n)_{n \in \mathbb{N}}$ is now a sequence of weights. The key remark of [BK, Theorem 4.1] is that

$$\mathcal{A}_G = P \mathcal{A}_G P \text{ and } \widetilde{\mathcal{A}_G P f}(|x|) = s_{|x|-1} \tilde{P} f(|x| - 1) + s_{|x|+1} \tilde{P} f(|x| + 1),$$

for all $f \in \mathcal{C}_c(V)$ with the convention that $s_{-1} = 0$. Using the unitary transformation $U: \ell^2(\mathbb{N}) \rightarrow \ell^2(\mathbb{N}, s_n)$ given by $U \tilde{f}(n) = \sqrt{s_n} \tilde{f}(n)$, we see that \mathcal{A}_G is unitarily equivalent to the direct sum of 0 and of a Jacobi matrix acting on $\ell^2(\mathbb{N})$, with 0 on the diagonal and the sequence $(\sqrt{s_n} \sqrt{s_{n+1}})_{n \in \mathbb{N}}$ on the off-diagonal. Finally using [Ber, Page 504 and 507], we derive that: Given $\alpha > 0$ and $s_n := \lfloor n^\alpha \rfloor$,

$$\eta(\mathcal{A}_G) = \begin{cases} 0, & \text{if } \alpha \leq 1, \\ 1, & \text{if } \alpha > 1. \end{cases}$$

Notation: The set of nonpositive integers is denoted by \mathbb{N} , note that $0 \in \mathbb{N}$. Given a set X and $Y \subseteq X$ let $\mathbf{1}_Y$ be the multiplication operator by the characteristic function of Y .

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REFERENCES

- [Ber] J.M. Berezanskii: *Expansions in eigenfunctions of selfadjoint operators*, Providence, R.I.: American Mathematical Society 1968. IX, 809 p. (1968).
- [BK] J. Breuer and M. Keller: *Spectral Analysis of Certain Spherically Homogeneous Graphs*, preprint arXiv:1201.0118v1

- [GS] S. Golénia and C. Schumacher: *The problem of deficiency indices for discrete Schrödinger operators on locally finite graphs*, J. Math. Phys. 52 (2011), no. 6, 063512, 17 pp.
- [MO] B. Mohar and M. Omladic: *The spectrum of infinite graphs with bounded vertex degrees*, Graphs, hypergraphs and applications, Teubner Texte 73 (Teubner, Leipzig, 1985), pp. 122–125.
- [MW] B. Mohar and W. Woess: *A survey on spectra of infinite graphs*, J. Bull. Lond. Math. Soc. **21**, No.3, 209-234 (1989).
- [Mü] V. Müller: *On the spectrum of an infinite graph*, Linear Algebra Appl. 93 (1987) 187–189.
- [RS] M. Reed and B. Simon: *Methods of Modern Mathematical Physics, Tome I–IV: Analysis of operators* Academic Press.

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